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Gauge symmetry as symmetry of matrix coordinates

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Abstract. We propose a new point of view on gauge theories, based on taking the action of symmetry transformations directly on the space coordinates. Via this approach the gauge fields are not introduced at the first step, and they can be interpreted as fluctuations around some classical solutions of the model. The new point of view is connected to the lattice formulation of gauge theories, and the parameter of the non-commutativity of the coordinates appears as the lattice spacing parameter. Through the statements concerning the continuum limit of lattice gauge theories, the suggestion arises that the non-commutative spaces are the natural ones to formulate gauge theories at strong coupling. Via this point of view, a close relation between the large-N limit of gauge theories and string theory can be made manifest.

Recently much attention has been paid to the formulation and study of field theories on non-commutative (NC) spaces [1–3]. Apart from the abstract mathematical interests, there were various physical motivations for doing so. One of the original motivations has been to get "finite" field theories via the intrinsic regularizations which are encoded in some of NC spaces [4,5]. The other motivation came from the unification aspects of theories on NC spaces. These unification aspects have been the result of the "algebraization" of "space, geometry and their symmetries" via the approach of NC geometry [6]. Interpreting the Higgs field as a gauge field in the discrete direction of a two-sheet space [7] and unifying gauge theories with gravity [8,9] are examples of this point of view on NC spaces.

The other motivation refers to the natural appearance of NC spaces in some areas of physics, and the recent one in string theory. It has been understood that string theory is involved by some kinds of non-commutativities; two examples are

(1) the coordinates of bound states of N D-branes are represented by $N \times N$ Hermitian matrices [10], and
(2) the longitudinal directions of D-branes in the presence of a B-field background appear to be NC, as seen by the ends of open strings [11,12,1].

As mentioned above, one of the motivations to formulate theories on NC spaces has been a unified treatment of the symmetries living in a space and the space itself. One of the most important symmetries in physical theories is gauge symmetry, and to be extreme in identifying the space with its symmetries is to take the action of symmetry transformations on the space. In usual gauge theories the action of the symmetry transformations is defined on the gauge fields, A^μ, but in the new picture one takes the action on space, and to be more specific, on the "coordinates" of space. It will be our main strategy in presenting a new point of view on gauge theories. As will be made clear later, the main tools and points of view for different subjects and discussions here are developed and originate in the D-branes of string theories [13,14]. Here we try to reorganize the facts and discussions to present a new picture for gauge theories and see how things should be by this approach. The action which we are concerned with here is the Eguchi–Kawai one [15], but with a different interpretation of the configurations which are described by the action. As we will see, the new interpretation is sufficiently rich to recover some aspects of gauge theories which were already known, but maybe as disjoint facts. It will be shown that the new interpretation is related on one side to the lattice formulation of gauge theories [16], and with a different representation is connected to the ordinary formulation of gauge theories. In relation with lattice gauge theory the parameter of the non-commutativity of the coordinates appears as the lattice spacing parameter. Through the statements concerning the continuum limit of lattice gauge theories the suggestion arises that NC spaces are the natural ones to formulate gauge theories at the strong coupling limit. Also the model can make manifest the close relation between the large-N limit of gauge theories, known to be the theory of "Feynman graphs" as the world-sheet of strings, and string theory [17].

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1 The model

As mentioned above, instead of introducing gauge fields, we define the gauge symmetry transformations directly on the generators of the displacement in space, calling them "coordinates" and representing them by $\hat{X}^\mu$ [3][1], and we assume them to be $N \times N$ Hermitian matrices. So to describe the generators in an infinite volume these matrices should be taken with $N \to \infty$, even when they are used to formulate a finite group gauge theory. So we take the definition of the gauge transformations as

$$\hat{X}^\mu \to \hat{X}^{\mu\prime} = \omega \hat{X}^\mu \omega^\dagger, \quad \mu = 1, \cdots, d, \tag{1}$$

where ω is an arbitrary unitary $N \times N$ matrix (so it belongs to a group, say G). This transformation is the same as in [3] but not in the infinitesimal form. On the other hand, it is the same transformation as the one acting on the coordinates of D-branes via $N \times N$ Hermitian matrices (see e.g. [18,19]). So the coordinates in a space which contains the bound states of N D-branes enjoy such a transformation. Also if from the beginning one chooses the matrices $\hat{X}^\mu$ to belong to $L^2_\infty(\boldsymbol{R}^d) \otimes M_{n\times n}$ in the form $\hat{X}^\mu = \mathrm{i}\partial^\mu \otimes \mathbf{1}_n + g_{_{YM}} 1 \otimes A^\mu$ [20], they will have the same behavior under gauge transformations as (1). So we are not very far from the usual language of gauge theories.

Our coordinates are matrices and NC, and as usual they are accompanied by a length scale which is the size of the NC effects to appear. Here we denote this length scale by ℓ. We define the unitary matrices

$$U_\mu \equiv \mathrm{e}^{\mathrm{i}\ell \hat{X}^\mu}, \tag{2}$$

as the operators which, acting on the states, constitute the displacement ℓ. With the ideas coming from lattice gauge theory, and also recalling the role of covariant derivatives as the tools of parallel transformations, we define the objects:

$$\Omega_{\mu\nu} \equiv U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger, \tag{3}$$

with the property $\Omega_{\mu\nu}^{-1} = \Omega_{\nu\mu} = \Omega_{\mu\nu}^\dagger$. Then the action of the model we take to be

$$S = -\frac{1}{g^2} \sum_{\mu,\nu} \mathrm{Tr}\,\Omega_{\mu\nu}, \tag{4}$$

which because of the Tr is invariant under the transformation (1). This action is essentially the Eguchi–Kawai one [15]. In the context of the Eguchi–Kawai model the symmetry of the action is a global symmetry, i.e. the symmetry transformations on the gauge fields are space independent. But as we will see, interpreting the $\hat{X}^\mu$s as space coordinates encodes a sufficiently rich structure in the model to extract gauge fields and their local symmetry transformations just as in the usual formulation of gauge theories. One may define in analogy with the field strength

$$\Omega_{\mu\nu} \equiv \mathrm{e}^{-\mathrm{i}\ell^2 F_{\mu\nu}}, \tag{5}$$

which in the small ℓ limit takes the form

$$F^{\mu\nu} = -\mathrm{i}\left[\hat{X}^\mu, \hat{X}^\nu\right] + \frac{1}{2}\ell\left[\hat{X}^\mu + \hat{X}^\nu, \left[\hat{X}^\mu, \hat{X}^\nu\right]\right] + O(\ell^2). \tag{6}$$

The action for small ℓ has the form

$$S|_{\ell\to 0} = -\frac{1}{g^2}\sum_{\mu,\nu} \mathrm{Tr}\left(1 - \mathrm{i}\ell^2 F_{\mu\nu} - \frac{1}{2}\ell^4 F_{\mu\nu}^2 + \cdots\right). \tag{7}$$

The linear term in $F_{\mu\nu}$ does not have a contribution to the action because it is antisymmetric in $\mu\nu$[2]. So for small values of ℓ we have

$$S|_{\ell\to 0} = -\frac{1}{2g^2}\ell^4 \sum_{\mu,\nu} \mathrm{Tr}\left[\hat{X}^\mu, \hat{X}^\nu\right]^2 + \text{const. term} + O(\ell^5). \tag{8}$$

The actions (4) or (8) are actions for the matrices describing the space and its symmetries. Issues such as the dynamical generation of space and its dimension, and also the gauge group via matrix theory have been discussed in [21,22].

2 Relation to lattice gauge theory (strong coupling)

The model described with the action (4) already has the form of the lattice gauge theory at large N, called the Eguchi–Kawai model. Here we also want to mention the connection to lattice gauge theory for finite groups. In fact the relation between NC geometry and also NC differential geometry with lattice gauge theory has already been established in previous works [23,24]. Here we try to construct the relation explicitly. Let us have a look at the action of lattice gauge theory:

$$S_{\mathrm{lgt}} = -\frac{1}{g^2}\sum_{\mu,\nu}\sum_{\mathbf{i}} \mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}aA^\mu_{\mathbf{i}}}\mathrm{e}^{\mathrm{i}aA^\nu_{\mathbf{i}+\mu}}\mathrm{e}^{-\mathrm{i}aA^\mu_{\mathbf{i}+\nu}}\mathrm{e}^{-\mathrm{i}aA^\nu_{\mathbf{i}}}\right), \tag{9}$$

with a the lattice spacing parameter and $\mathbf{i}$ as a d-vector representing a site in the d dimensional lattice. Also we have used the symbol $\mathbf{i}+\mu$ for $(i_1, \cdots, i_\mu + 1, \cdots, i_d)$. To get a $U(m)$ lattice gauge theory, as the first step, take the Ansatz resulting from d times block-diagonalizations of the matrices $\hat{X}^\mu$, with the size of the last block $m \times m$. So the action takes the form

$$S_{\mathrm{blocked}} = -\frac{1}{g^2}\sum_{\mu,\nu}\sum_{\mathbf{i}} \mathrm{Tr}\left(\mathrm{e}^{\mathrm{i}\ell\hat{x}^\mu_{\mathbf{i}}}\mathrm{e}^{\mathrm{i}\ell\hat{x}^\nu_{\mathbf{i}}}\mathrm{e}^{-\mathrm{i}\ell\hat{x}^\mu_{\mathbf{i}}}\mathrm{e}^{-\mathrm{i}\ell\hat{x}^\nu_{\mathbf{i}}}\right), \tag{10}$$

in which the index i_j in the vector $\mathbf{i}$ is counting the place of a block in the jth step of the block-diagonalizations. The Tr above is for the $U(m)$ structure of the $\hat{x}^\mu_{\mathbf{i}}$ matrices. But

[1] In [3] these objects are called "covariant coordinates"

[2] It is not true that because of Tr the linear term can be ignored. For infinite dimensional matrices one can get a nonzero trace from a commutator

this action is still different from the lattice action (9). To make the exact correspondence we should do a slight modification in one of the steps of the block-diagonalizations. Firstly, take the matrix Δ as

$$\begin{aligned} &\Delta_{rs} = \delta_{r,s-1}, \quad \text{for infinite size,} \\ &\Delta_{rs} = \delta_{r,s-1}, \quad \Delta_{p1} = 1, \quad \text{for size } p \times p, \end{aligned} \tag{11}$$

with the properties $\Delta^{-1} = \Delta^{\mathrm{T}} = \Delta^{\dagger}$, so Δ is unitary. For this matrix and a diagonal matrix A we have

$$\Delta \text{diag.}(a_1, \cdots, a_{p-1}, a_p) \Delta^{-1} = \text{diag.}(a_2, \cdots, a_p, a_1). \tag{12}$$

By using the matrix Δ we modify the block-diagonalizations mentioned above, by requiring that in the μth step of the diagonalizations of the matrix $\hat{X}^{\mu}$ it picks up a Δ with the appropriate size, as in

$$\hat{X}^{\mu} \underset{\mu\text{th step}}{\longrightarrow} \hat{x}^{\mu} \Delta. \tag{13}$$

So in two steps of d steps two pairs of Δ and Δ^{-1} appear around the $\hat{X}^{\mu}$ and $\hat{X}^{\nu}$ matrices in the action, and this causes the appropriate shift in the blocks to obtain the action of lattice gauge theory (9). In comparison with the lattice action one sees that the parameter ℓ has appeared as the lattice spacing parameter. This means that the lattice spacing parameter is a measure of the appearance of NC effects [24]. Based on the lattice calculations, one can derive the relation between the parameters ℓ, the coupling constant g and the string tension K, and via this relation the statement follows that the continuum limits of the lattice gauge theories are obtained just at exactly zero coupling [19,25]. So the suggestion arises that the strong coupling limit of the gauge theories will find a reasonable and natural formulation in NC spaces (for more discussions of this point see [19,18]). Also via this explicit construction the observation is made that both the structure of space (here a lattice) and also the gauge fields living in the space can be extracted from the large matrices $\hat{X}^{\mu}$. We see another example of this behavior in the relation between the model and the ordinary formulation of gauge theories.

3 Relation to ordinary gauge theory (weak coupling)

It is known that the classical action of lattice gauge theories at small lattice parameter is equivalent with the classical action of gauge theories, the so-called weak coupling limit of lattice gauge theory [25]. So up to now, by taking the limit $\ell \to 0$ in the action obtained in the previous section we can get the ordinary action of gauge theories. In the following we give another presentation of this, which of course contains the procedure of going to the continuum limit, but a little implicitly. To get the ordinary gauge theory we use the techniques which have been developed in constructing D-branes from matrix theories [26,27]. Here we just recall the construction and refer the reader to the literature (see e.g. [28]). For large matrices one always can find a set of matrix pairs $(\hat{q}^i, \hat{p}^i)$ with sizes $n_i \times n_i$ so that

$$\left[\hat{q}^i, \hat{p}^j\right] = \mathrm{i}\delta_{ij} \mathbf{1}_{n_i}. \tag{14}$$

The above commutator is not satisfied for finite dimensional matrices. We assume that the eigenvalues of $\hat{q}^i$ and $\hat{p}^i$ are distributed uniformly in the interval $[0, (2\pi n_i)^{1/2}]$. To get a $U(m)$ gauge theory one can break the matrices $\hat{X}^{\mu}$ with size N down to matrices with sizes n_i and m such that $N = m \cdot n_1 n_2 \cdots n_{d/2}$ when d is even, and $N = m \cdot n_1 n_2 \cdots n_{(d+1)/2}$ for d odd, with the condition $N, n_i \to \infty$ and m finite. On the other hand, it is easy to see that matrices in the form

$$\begin{aligned} \ell^2 \hat{X}_{\text{cl}}^{2i-1} &= \underbrace{\mathbf{1}_{n_1} \otimes \cdots}_{i-1} \frac{\hat{q}^i L_i}{\sqrt{2\pi n_i}} \otimes \cdots \otimes \mathbf{1}_{n_{d/2}} \otimes \mathbf{1}_m, \\ \ell^2 \hat{X}_{\text{cl}}^{2i} &= \underbrace{\mathbf{1}_{n_1} \otimes \cdots}_{i-1} \frac{\hat{p}^i L_{i+1}}{\sqrt{2\pi n_i}} \otimes \cdots \otimes \mathbf{1}_{n_{d/2}} \otimes \mathbf{1}_m, \\ & i = 1, \cdots, d/2, \end{aligned} \tag{15}$$

for even d, and with an extra one

$$\ell^2 \hat{X}_{\text{cl}}^{d} = \underbrace{\mathbf{1}_{n_1} \otimes \cdots}_{\frac{d-1}{2}} \frac{\hat{q}^{\frac{d+1}{2}} L_d}{\sqrt{2\pi n_{\frac{d+1}{2}}}} \otimes \mathbf{1}_m, \tag{16}$$

for odd d, solve the equations of motion derived from the action. Here the L_is have the interpretation of large radii of compactifications [26,27]. By the equations of motion for n_i one obtains [27]

$$\frac{L_i L_{i+1}}{2\pi n_i} \sim \ell^2. \tag{17}$$

By admitting fluctuations around the classical solutions, one can write

$$\hat{X}^{\mu} = \hat{X}_{\text{cl}}^{\mu} + g_{_{YM}} A^{\mu}, \tag{18}$$

where the A_{μ} are $N \times N$ Hermitian matrices and functions of the $(\hat{q}^i, \hat{p}^i)$ matrices, also with the same structure of the matrices $\hat{X}_{\mu}$. By inserting $\hat{X}_{\mu}$ and expanding the action in the $\ell \to 0$ limit up to second order of the fluctuations, and with the identifications [28,27,26]

$$\begin{aligned} &[\hat{p}_i, *] \sim \mathrm{i}\partial_{2i-1} *, \\ &[\hat{q}_i, *] \sim \mathrm{i}\partial_{2i} *, \\ &\text{Tr}(\cdots) \to \int \mathrm{d}^d x (\cdots), \end{aligned} \tag{19}$$

one recovers the ordinary action for $U(m)$ gauge theory. The coupling constant of the resulting gauge theory is found to be $g_{_{YM}}^2 \sim \ell^{d-4} g^2$, which shows that in the limit of small ℓ and for $d \geq 4$ the theory corresponds to the weak coupling limit.

4 Large-N gauge theory and string theory

It is known that in a diagrammatic representation, the partition function of a gauge theory at large N is given by the $(1/N)^{\text{genus}}$ expansion, with genus taken to be that of the "big" Feynman graphs of the theory. Also it is shown that the density of "holes" (quark loops) in the graphs goes to zero with $1/N$. So in the extreme large-N limit the theory is described by smooth graphs. By interpreting $1/N$ as the coupling constant of a string theory, the expansion mentioned above takes the form of the standard string perturbation one [17]. All of the features mentioned here can be described by the point of view proposed in this work. Firstly, at large N the action can take the form of that of free strings. We are thinking of smooth strings, so we take $\ell \to 0$ and $N \to \infty$. Thus the action becomes

$$S|_{\ell\to 0} = -\frac{1}{2\ell^4 g^2}\sum_{\mu,\nu} \mathrm{Tr}\left[\hat{X}^\mu, \hat{X}^\nu\right]^2 - \frac{1}{g^2}\mathrm{Tr}\mathbf{1}_N, \quad (20)$$

in which we have applied the replacement $\hat{X}^\mu \to \hat{X}^\mu/\ell^2$ and so the new $\hat{X}^\mu$ has the length dimension. To get free strings one uses the map between the matrix variables $(\hat{q},\hat{p})$ and the continuous phase space variables (σ_1,σ_2) [27,26,29]

$$\begin{aligned} &\mathrm{Tr}(\cdots) \to \int \mathrm{d}^2\sigma \sqrt{\det g_{rs}}(\cdots),\\ &[A,B] \to \{A,B\}_{\mathrm{PB}},\\ &[\hat{q},\hat{p}] = \mathrm{i} \to \{\sigma_1,\sigma_2\}_{\mathrm{PB}} = 1/\sqrt{\det g_{rs}},\\ &[\hat{p},*] \to \mathrm{i}\partial_1 *, \quad [\hat{q},*] \to \mathrm{i}\partial_2 *, \end{aligned} \quad (21)$$

with the definition of the Poisson bracket $\{A,B\}_{\mathrm{PB}} = 1/(\det g_{rs})^{1/2}\epsilon^{rs}\partial_r A \partial_r B$, $(r,s=1,2)$. By these replacements one gets the action of free strings in the Schild form [30,27]. Also by solving the equation of motion for $(\det g_{rs})^{1/2}$ and inserting the solution in the action one can obtain the Nambu–Goto action.

The issue of interactions is more subtle, and it also has been approached previously [31]. It is shown that the $1/N$ expansion for this action corresponds to the perturbation theory of strings by reproducing the light-cone string field theory through the Schwinger–Dyson equations.

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